TOPOLOGICAL STRUCTURES OF HYPERSPACES OF FINITE SETS IN NON-SEPARABLE METRIZABLE SPACES

KATSUHISA KOSHINO

Abstract. Let $\mathrm{Fin}(X)$ be the hyperspace consisting of non-empty finite subsets of a space X endowed with the Vietoris topology. In this paper, we characterize a metrizable space X whose hyperspace $\mathrm{Fin}(X)$ is homeomorphic to the linear subspace spanned by the canonical orthonormal basis of a non-separable Hilbert space.

1. Introduction

Throughout this paper, spaces are metrizable, maps are continuous and κ is an infinite cardinal. By $\mathrm{Fin}(X)$, we denote the hyperspace of non-empty finite sets in a space X endowed with the Vietoris topology. Let $\ell_2(\kappa)$ be the Hilbert space of density κ and $\ell_2^f(\kappa)$ be the linear subspace spanned by the canonical orthonormal basis of $\ell_2(\kappa)$. Topological structures of hyperspaces are classical subjects and have been studied in infinite-dimensional topology. In 1985, D.W. Curtis and N.T. Nhu [4] characterized a space X whose hyperspace $\mathrm{Fin}(X)$ is homeomorphic to $\ell_2^f(\omega)$ as follows:

Theorem 1.1. *A space X is non-degenerate, connected, locally path-connected, strongly countable-dimensional*[1] *and σ-compact*[2] *if and only if $\mathrm{Fin}(X)$ is homeomorphic to $\ell_2^f(\omega)$.*

By an X-*manifold*, we mean a topological manifold modeled on a space X. In the general case, K. Mine, K. Sakai and M. Yaguchi [7] proved the following:

Theorem 1.2. *For a connected $\ell_2^f(\kappa)$-manifold X, the hyperspace $\mathrm{Fin}(X)$ is homeomorphic to $\ell_2^f(\kappa)$.*

In this paper, we shall try to generalize their results as follows:

Main Theorem. *A space X is connected, locally path-connected, strongly countable-dimensional, σ-locally compact and for each point $x \in X$, any neighborhood of x in X is of density κ if and only if $\mathrm{Fin}(X)$ is homeomorphic to $\ell_2^f(\kappa)$.*

2. Preliminaries

In this section, we will fix some notation and introduce a characterization of $\ell_2^f(\kappa)$-manifolds used in the proof of the main theorem. Moreover, we show some lemmas concerning subdivisions of simplicial complexes that will be needed in the sequel. The closed unit interval $[0,1]$ is denoted by $\mathbf{I}$. Let $X = (X,d)$ be a metric space. For a point $x \in X$ and subsets $A, B \subset X$, we put

2010 *Mathematics Subject Classification.* Primary: 54B20, Secondary: 54F65, 57N20.

Key words and phrases. hyperspace, the Vietoris topology, the Hausdorff metric, Hilbert space, Z-set, Strong Z-set, strong universality, discrete cells property.

This work was supported by JSPS KAKENHI Grant Number 15K17530.

[1] A space X is strongly countable-dimensional if it is written as a countable union of finite-dimensional closed subsets.

[2] A space X is called to be σ-(locally)compact if it is a countable union of (locally)compact subsets.

$d(x,A) = \inf\{d(x,a) \mid a \in A\}$ and $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}$. For $\epsilon > 0$, we define $B_d(x,\epsilon) = \{x' \in X \mid d(x,x') < \epsilon\}$, $\overline{B_d}(x,\epsilon) = \{x' \in X \mid d(x,x') \leq \epsilon\}$, $N_d(A,\epsilon) = \{x' \in X \mid d(x',A) < \epsilon\}$ and $\overline{N_d}(A,\epsilon) = \{x' \in X \mid d(x',A) \leq \epsilon\}$. By $\operatorname{diam}_d A$, we mean the diameter of A. It is well-known that the topology of $\operatorname{Fin}(X)$ coincides with the one induced by *the Hausdorff metric* d_H defined as follows:

$$d_H(A,B) = \inf\{r > 0 \mid A \subset N_d(B,r), B \subset N_d(A,r)\}.$$

For maps $f : X \to Y$ and $g : X \to Y$, and for an open cover $\mathcal{U}$ of Y, f is *$\mathcal{U}$-close* to g if for each point $x \in X$, there is a member $U \in \mathcal{U}$ that contains the both $f(x)$ and $g(x)$. A closed subset A of a space X is called to be a *(strong) Z-set* in X provided that for each open cover $\mathcal{U}$ of X, there is a map $f : X \to X$ such that f is $\mathcal{U}$-close to the identity map on X and the (closure of)image $f(X)$ misses A. These concepts play important roles in infinite-dimensional topology. A *Z-embedding* means an embedding whose image is a Z-set. We say that a space X is *strongly universal* for a class $\mathcal{C}$ if the following condition holds:

- For each space $A \in \mathcal{C}$, each closed subset B of A, each open cover $\mathcal{U}$ of X, and each map $f : A \to X$ such that the restriction $f|_B$ is a Z-embedding, there exists a Z-embedding $g : A \to X$ such that g is $\mathcal{U}$-close to f and $g|_B = f|_B$.

A space X has *the κ-discrete n-cells property*, $n < \omega$, if the following condition is satisfied:

- For every open cover $\mathcal{U}$ of X and every map $f : \bigoplus_{\gamma<\kappa} A_\gamma \to X$, where each $A_\gamma = \mathbf{I}^n$, there exists a map $g : \bigoplus_{\gamma<\kappa} A_\gamma \to X$ such that g is $\mathcal{U}$-close to f and the family $\{g(A_\gamma)\}_{\gamma<\kappa}$ is discrete in X.

The author [5] gave the following characterization to an $\ell_2^f(\kappa)$-manifold:

Theorem 2.1. *A connected space X is an $\ell_2^f(\kappa)$-manifold if and only if the following conditions are satisfied:*

(1) *X is a strongly countable-dimensional, σ-locally compact ANR of density κ;*
(2) *X is strongly universal for the class of finite-dimensional compact metrizable spaces;*
(3) *every finite-dimensional compact subset of X is a strong Z-set in X;*
(4) *X has the κ-discrete n-cells property for every $n < \omega$.*

In the above theorem, replacing "ANR" with "AR", we have a characterization of the model space $\ell_2^f(\kappa)$. Given a simplicial complex K, we denote the polyhedron[3] of K by $|K|$ and the n-skeleton of K by $K^{(n)}$ for each $n < \omega$. Regarding $\sigma \in K$ as a simplicial complex consisting of its faces, we write $\sigma^{(n)}$ as the union of i-faces of σ, $i \leq n$. The next two lemmas are used in the proof of Theorem E of [2].

Lemma 2.2. *Let $Y = (Y,\rho)$ be a metric space, K a simplicial complex and $f : |K| \to Y$ a map. For each map $\alpha : Y \to (0,\infty)$, there exists a subdivision K' of K such that $\operatorname{diam}_\rho f(\sigma) < \inf_{x\in\sigma} \alpha f(x)$ for all $\sigma \in K'$.*

Proof. Let $\alpha : Y \to (0,\infty)$ be a map. By the continuity of α, we can find $0 < \delta(y) \leq \alpha(y)/4$ for each $y \in Y$ such that for every $y' \in B_\rho(y,\delta(y))$, $\alpha(y') \geq \alpha(y)/2$. It follows from [8, Theorem 4.7.11], there is a subdivision K' of K that refines the open cover $\{f^{-1}(B_\rho(y,\delta(y))) \mid y \in Y\}$. To show that K' is the desired subdivision, take any $\sigma \in K'$. Then we can choose $y \in Y$ so that $f(\sigma) \subset B_\rho(y,\delta(y))$, and hence $\operatorname{diam}_\rho f(\sigma) < 2\delta(y) \leq \alpha(y)/2$. Observe that for each $x \in \sigma$, $\alpha f(x) \geq \alpha(y)/2$, which implies that $\operatorname{diam}_\rho f(\sigma) < \inf_{x\in\sigma} \alpha f(x)$. Thus the proof is complete. □

[3]In this paper, polyhedra are not needed to be metrizable.

Lemma 2.3. *For each map $\alpha : |K| \to (0,\infty)$ of the polyhedron of a simplicial complex K and $\beta > 1$, there is a subdivision K' of K such that $\sup_{x\in\sigma} \alpha(x) < \beta \inf_{x\in\sigma} \alpha(x)$ for any $\sigma \in K'$.*

Proof. For each $x \in |K|$, we can choose an open neighborhood $U(x)$ of x in $|K|$ so that if $y \in U(x)$, then $|\alpha(x) - \alpha(y)| < (\beta - 1)\alpha(x)/(\beta + 1)$. Then $\mathcal{U} = \{U(x) \mid x \in |K|\}$ is an open cover of $|K|$. According to Theorem 4.7.11 of [8], there is a subdivision K' of K that refines $\mathcal{U}$. Take any simplex $\sigma \in K'$ and any point $y \in \sigma$. By the compactness of σ, we can find $z \in \sigma$ such that $\alpha(z) = \inf_{z'\in\sigma} \alpha(z')$. Since K' refines $\mathcal{U}$, there exists a point $x \in |K|$ such that $\sigma \subset U(x)$. Then $|\alpha(x) - \alpha(y)| < (\beta - 1)\alpha(x)/(\beta + 1)$ and $|\alpha(x) - \alpha(z)| < (\beta - 1)\alpha(x)/(\beta + 1)$. Observe that $\alpha(y) < 2\beta\alpha(x)/(\beta + 1) < \beta\alpha(z)$, which implies that $\sup_{z'\in\sigma} \alpha(z') < \beta \inf_{z'\in\sigma} \alpha(z')$. Hence K' is the desired subdivision. □

3. Basic properties of $\mathrm{Fin}(X)$

In this section, we list some basic properties of the hyperspace $\mathrm{Fin}(X)$. According to Proposition 5.3 of [7], we have the following:

Proposition 3.1. *A space X is strongly countable-dimensional, σ-locally compact and of density κ if and only if so is $\mathrm{Fin}(X)$.*

Proposition 3.2. *Let X be a space. For each point $x \in X$, if every neighborhood of $\{x\}$ in $\mathrm{Fin}(X)$ is of density κ, then so is any neighborhood of x in X.*

Proof. Since $\mathrm{Fin}(X)$ is of density κ, so is X due to Proposition 3.1. Let $x \in X$ be a point such that any neighborhood of $\{x\}$ in $\mathrm{Fin}(X)$ is of density κ. Assume that there is a neighborhood U of x whose density $< \kappa$. As is easily observed, $\mathrm{Fin}(U)$ is a neighborhood of $\{x\}$ in $\mathrm{Fin}(X)$. Applying Proposition 3.1 again, we have that the density of $\mathrm{Fin}(U)$ is less than κ, which is a contradiction. Consequently, every neighborhood of x is of density κ. □

Combining Lemmas 2.3 and 3.6 with the proof of Theorem 2.4 in [4] (cf. [9, Proposition 3.1]), we can establish the following:

Proposition 3.3. *A space X is locally path-connected (connected and locally path-connected) if and only if $\mathrm{Fin}(X)$ is an ANR (AR).*

4. The strong universality of $\mathrm{Fin}(X)$

This section is devoted to verifying the strong universality of $\mathrm{Fin}(X)$ for the class of finite-dimensional compact metrizable spaces. The following lemma follows from Lemma 2.2 of [4].

Lemma 4.1. *Let X be a space. If $\mathcal{A} \subset \mathrm{Fin}(X)$ is locally connected and compact, then so is the union $\bigcup \mathcal{A} \subset X$.*

Using Curtis and Nhu's result [4], we shall show the strong universality of a non-separable hyperspace $\mathrm{Fin}(X)$.

Proposition 4.2. *Let X be a non-degenerate, connected, locally path-connected space. Then $\mathrm{Fin}(X)$ is strongly universal for the class of finite-dimensional compact metrizable spaces.*

Proof. Let A be a finite-dimensional compact metrizable space, B a closed set in A, $f : A \to \mathrm{Fin}(X)$ a map such that $f|_B$ is an Z-embedding. We show that for any open cover $\mathcal{U}$ of $\mathrm{Fin}(X)$, there is a Z-embedding $g : A \to \mathrm{Fin}(X)$ such that g is $\mathcal{U}$-close to f and $g|_B = f|_B$. Since A is finite-dimensional and compact, we can regard it as a closed subset of $\mathbf{I}^n$ for some $n < \omega$. The space X is connected and locally path-connected, and hence $\mathrm{Fin}(X)$ is an AR by Proposition 3.3. Therefore

the map f can be extended to a map $\tilde{f} : \mathbf{I}^n \to \mathrm{Fin}(X)$. Note that $\mathcal{A} = \tilde{f}(\mathbf{I}^n)$ is connected, locally connected and compact. According to Lemma 4.1, the union $\bigcup \mathcal{A} \subset X$ is also locally connected and compact, and hence it is locally path-connected and has finitely many components. Note that each component of $\bigcup \mathcal{A}$ is open and closed. We may assume that at least one component of $\bigcup \mathcal{A}$ is non-degenerate because X has no isolated points and is locally path-connected. Let

$$\overline{\mathcal{A}} = \left\{ F \in \mathrm{Fin}(\bigcup \mathcal{A}) \;\middle|\; F \text{ contains some element of } \mathcal{A} \right\}.$$

As is easily observed, the space $\overline{\mathcal{A}}$ is separable and for each $F \in \mathrm{Fin}(\bigcup \mathcal{A})$, $F \in \overline{\mathcal{A}}$ if F contains some element of $\overline{\mathcal{A}}$.

We show that any $F \in \overline{\mathcal{A}}$ meets each component of $\bigcup \mathcal{A}$. It is sufficient to prove that every $F \in \mathcal{A}$ intersects each component of $\bigcup \mathcal{A}$. Suppose the contrary, so there are $F \in \mathcal{A}$ and a component C of $\bigcup \mathcal{A}$ such that $F \cap C = \emptyset$. Then C and $\bigcup \mathcal{A} \setminus C$ are open and $F \subset \bigcup \mathcal{A} \setminus C$. Observe that $\{F' \in \mathcal{A} \mid F' \cap C \neq \emptyset\}$ and $\{F' \in \mathcal{A} \mid F' \subset \bigcup \mathcal{A} \setminus C\}$ are non-empty disjoint open sets in $\mathcal{A}$. Moreover,

$$\mathcal{A} = \left\{ F' \in \mathcal{A} \;\middle|\; F' \cap C \neq \emptyset \right\} \cup \left\{ F' \in \mathcal{A} \;\middle|\; F' \subset \bigcup \mathcal{A} \setminus C \right\}.$$

This contradicts to the connectedness of $\mathcal{A}$. Hence any $F \in \overline{\mathcal{A}}$ meets all components of $\bigcup \mathcal{A}$. Applying Lemma 4.6 of [4], we have that $\overline{\mathcal{A}}$ is strongly universal for the class of finite-dimensional compact metrizable spaces. Therefore there exists an embedding $g : A \to \overline{\mathcal{A}} \subset \mathrm{Fin}(X)$ such that g is $\mathcal{U}$-close to f and $g|_B = f|_B$. By Lemma 3.8 of [4], the compact image $g(A)$ is a Z-set in $\mathrm{Fin}(X)$. The proof is completed. □

5. The strong Z-set property of compact sets in $\mathrm{Fin}(X)$

In this section, we will discuss the strong Z-set property of compact subsets of $\mathrm{Fin}(X)$.

Lemma 5.1. *If a metric space $X = (X, d)$ is non-degenerate and connected, then for each $x \in X$ and $0 < \epsilon < \mathrm{diam}_d X/2$, there exists a point $y \in X$ such that $d(x, y) = \epsilon$.*

Proof. Suppose the contrary. Then X can be separated by disjoint non-empty open subsets $B_d(x, \epsilon)$ and $X \setminus \overline{B_d}(x, \epsilon)$, which contradicts to the connectedness of X. The proof is complete. □

Let $\mathrm{Comp}(X)$ be the hyperspace of non-empty compact sets in a space X with the Vietoris topology. Note that $\mathrm{Fin}(X) \subset \mathrm{Comp}(X)$. The analogues of the following two lemmas for $\mathrm{Comp}(X)$ are used in the proof of Theorem E of [2].

Lemma 5.2. *Let $X = (X, d)$ be a metric space. Suppose that $\{A_n\}_{n<\omega}$ is a sequence in $\mathrm{Fin}(X) = (\mathrm{Fin}(X), d_H)$ converging to $A \in \mathrm{Fin}(X)$. Then for each $B_n \subset A_n$, $\{B_n\}_{n<\omega}$ has a subsequence converging to some $B \subset A$.*

Proof. According to Lemma 1.11.2. (3)[4] of [6], $\tilde{A} = A \cup \bigcup_{n<\omega} A_n$ is compact. Hence the hyperspace $\mathrm{Comp}(\tilde{A}) = (\mathrm{Comp}(\tilde{A}), (d|_{\tilde{A}\times\tilde{A}})_H)$ is compact, see [8, Theorem 5.12.5. (3)], which implies that $\{B_n\}_{n<\omega}$ has a subsequence $\{B_{n_i}\}_{i<\omega}$ converging to some $B \in \mathrm{Comp}(\tilde{A})$. By Lemma 1.11.2. (2)[4] of [6], we have

$$\begin{aligned} B &= \{x \in X \mid \text{for each } i < \omega, \text{ there is } b_{n_i} \in B_{n_i} \text{ such that } \lim_{i\to\infty} b_{n_i} = x\} \\ &\subset \{x \in X \mid \text{for each } i < \omega, \text{ there is } a_{n_i} \in A_{n_i} \text{ such that } \lim_{i\to\infty} a_{n_i} = x\} = A. \end{aligned}$$

Thus the proof is complete. □

[4]This holds without the assumption that X is separable.

Lemma 5.3. *Let $X = (X, d)$ be a metric space and $\alpha : \mathrm{Fin}(X) \to (0, \infty)$ be a map. If X is locally path-connected, then there exists a map $\beta : \mathrm{Fin}(X) \to (0, \infty)$ such that for any $A \in \mathrm{Fin}(X)$, each point $x \in \overline{N_d}(A, \beta(A))$ has an arc $\gamma : \mathbf{I} \to X$ from some point of A to x of $\mathrm{diam}_d\, \gamma(\mathbf{I}) < \alpha(A)$.*

Proof. For each $A \in \mathrm{Fin}(X)$, let

$$\Xi(A) = \left\{ \eta > 0 \;\middle|\; \begin{array}{l} \text{there exists } 0 < \epsilon < \alpha(A) \text{ such that for any } a \in A \text{ and } x \in \overline{B_d}(a, \eta), \\ \text{there is an arc from } a \text{ to } x \text{ of diameter } < \epsilon \end{array} \right\}$$

and $\xi(A) = \sup \Xi(A)$. Note that $\Xi(A) \neq \emptyset$ for all $A \in \mathrm{Fin}(X)$. Indeed, let $0 < \epsilon < \alpha(A)$. Since X is locally path-connected, and hence locally arcwise-connected [8, Corollary 5.14.7], for each $a \in A$, there exists $\eta(a) > 0$ such that for any $x \in \overline{B_d}(a, \eta(a))$, a and x are connected by an arc of diameter $< \epsilon$. Then $\eta = \min_{a \in A} \eta(a) \in \Xi(A)$. By the definition, $\xi(A) \leq \alpha(A)$.

We shall show that ξ is lower semi-continuous. Take any $t \in (0, \infty)$ and any $A \in \xi^{-1}((t, \infty))$. Then we can choose $t < \eta \leq \xi(A)$ so that there is $0 < \epsilon < \alpha(A)$ such that for any $a \in A$ and any $x \in \overline{B_d}(a, \eta)$, a and x are connected by an arc of diameter $< \epsilon$. Since X is locally arcwise-connected, there exists $\delta_1 > 0$ such that any $a \in A$ and any $x \in \overline{B_d}(a, \delta_1)$ are connected by an arc of diameter $< (\alpha(A) - \epsilon)/2$. By the continuity of α, we can find $\delta_2 > 0$ such that for each $B \in B_{d_H}(A, \delta_2)$, $|\alpha(A) - \alpha(B)| < (\alpha(A) - \epsilon)/2$. Let $\delta = \min\{\delta_1, \delta_2, (\eta - t)/2\}$ and $B \in B_{d_H}(A, \delta)$. Observe that $(\alpha(A) + \epsilon)/2 < \alpha(B)$. Fix any $b \in B$ and any $x \in \overline{B_d}(b, (\eta + t)/2)$. Since $d_H(A, B) < \delta$, we can take $a \in A$ such that $d(a, b) < \delta \leq \delta_1$, and hence there exists an arc γ_1 from b to a of diameter $< (\alpha(A) - \epsilon)/2$. On the other hand,

$$d(a, x) \leq d(a, b) + d(b, x) < \delta + (\eta + t)/2 \leq (\eta - t)/2 + (\eta + t)/2 = \eta,$$

which implies that there is an arc γ_2 from a to x of diameter $< \epsilon$. Joining these arcs γ_1 and γ_2, we can obtain an arc from b to x of diameter $< (\alpha(A) - \epsilon)/2 + \epsilon$, that is less than $\alpha(B)$. Hence $t < (\eta + t)/2 \leq \xi(B)$, which means that ξ is lower semi-continuous.

According to Theorem 2.7.6 of [8], we can find a map $\beta : \mathrm{Fin}(X) \to (0, \infty)$ such that $0 < \beta(A) < \xi(A)$ for all $A \in \mathrm{Fin}(X)$, that is the desired map. □

The next lemma is useful to detect a strong Z-set in an ANR.

Lemma 5.4 (Lemma 7.2 of [3])**.** *Let A be a topologically complete, closed subset of an ANR Y. If A is a countable union of strong Z-sets in Y, then it is a strong Z-set.*

We denote the cardinality of a set A by $\mathrm{card}\, A$. For each $k < \omega$, let $\mathrm{Fin}^k(X) = \{A \in \mathrm{Fin}(X) \mid \mathrm{card}\, A \leq k\}$. As is easily observed, $\mathrm{Fin}^k(X)$ is closed in $\mathrm{Fin}(X)$.

Proposition 5.5. *Suppose that X is non-degenerate, connected and locally path-connected. Then for each $k < \omega$, $\mathrm{Fin}^k(X)$ is a strong Z-set in $\mathrm{Fin}(X)$.*

Proof. Let $\mathcal{U}$ be an open cover of $\mathrm{Fin}(X)$ and $k < \omega$. We shall construct a map $\phi : \mathrm{Fin}(X) \to \mathrm{Fin}(X)$ so that ϕ is $\mathcal{U}$-close to the identity map on $\mathrm{Fin}(X)$ and $\mathrm{cl}\, \phi(\mathrm{Fin}(X)) \cap \mathrm{Fin}^k(X) = \emptyset$, where for a subset $\mathcal{A} \subset \mathrm{Fin}(X)$, $\mathrm{cl}\, \mathcal{A}$ means the closure of $\mathcal{A}$ in $\mathrm{Fin}(X)$. Take an open cover $\mathcal{V}$ of $\mathrm{Fin}(X)$ that is a star-refinement of $\mathcal{U}$. Since $\mathrm{Fin}(X)$ is an AR by Proposition 3.3, there are a simplicial complex K and maps $f : \mathrm{Fin}(X) \to |K|$, $g : |K| \to \mathrm{Fin}(X)$ such that gf is $\mathcal{V}$-close to the identity map on $\mathrm{Fin}(X)$, refer to [8, Theorem 6.6.2]. It remains to show that there exists a map $h : |K| \to \mathrm{Fin}(X)$ $\mathcal{V}$-close to g such that $\mathrm{cl}\, h(|K|) \cap \mathrm{Fin}^k(X) = \emptyset$ because $\phi = hf$ will be the desired map.

Fix any admissible metric d on X. Then we can take a map $\alpha : \mathrm{Fin}(X) \to (0, \min\{1, \mathrm{diam}_d\, X\})$ so that the collection $\{B_{d_H}(A, 2\alpha(A)) \mid A \in \mathrm{Fin}(X)\}$ refines $\mathcal{V}$. Since X is locally path-connected, due to Lemma 5.3, there is a map $\beta : \mathrm{Fin}(X) \to (0, \infty)$ such that for any $A \in \mathrm{Fin}(X)$, each point

$x \in \overline{N_d}(A, \beta(A))$ has an arc $\gamma : \mathbf{I} \to X$ from some point of A to x of $\operatorname{diam}_d \gamma(\mathbf{I}) < \alpha(A)/2$. We may assume that $\beta(A) \le \alpha(A)/2$ for every $A \in \operatorname{Fin}(X)$. Combining Lemmas 2.2 with 2.3, we can replace K with a subdivision so that for each $\sigma \in K$,

(1) $\operatorname{diam}_{d_H} g(\sigma) < \inf_{y\in\sigma} \beta g(y)/2$,
(2) $\sup_{y\in\sigma} \beta g(y) < 2 \inf_{y\in\sigma} \beta g(y)$,
(3) $\sup_{y\in\sigma} \alpha g(y) < 4 \inf_{y\in\sigma} \alpha g(y)/3$.

For every $v \in K^{(0)}$, fix a point $x(v) \in g(v)$. According to Lemma 5.1, we can find a point $z(v, j) \in X$ with $d(x(v), z(v, j)) = j\beta g(v)/(4(k + 1))$ for each $j = 0, \cdots, k$. Let $h(v) = g(v) \cup \{z(v, j) \mid j = 0, \cdots, k\}$. Clearly, $\operatorname{card} h(v) \ge k+1$ and $d_H(g(v), h(v)) \le \beta g(v) \le \alpha g(v)/2$. Observe that for any $0 \le i < j \le k$,

$$\begin{aligned} d(z(v, i), z(v, j)) &\ge |d(x(v), z(v, i)) - d(x(v), z(v, j))| = (j - i)\beta g(v)/(4(k + 1)) \\ &\ge \beta g(v)/(4(k + 1)). \end{aligned}$$

Next, we will extend h over $|K^{(1)}|$. Let $\sigma \in K^{(1)} \setminus K^{(0)}$, $\sigma^{(0)} = \{v_1, v_2\}$ and $\hat\sigma$ be the barycenter of σ. Due to conditions (1) and (2), we have for any $m = 1, 2$ and $j = 0, \cdots, k$,

$$\begin{aligned} d(z(v_m, j), g(\hat\sigma)) &\le d(z(v_m, j), g(v_m)) + d_H(g(v_m), g(\hat\sigma)) < \beta g(v_m)/4 + \operatorname{diam}_{d_H} g(\sigma) \\ &< \sup_{y\in\sigma} \beta g(y)/4 + \inf_{y\in\sigma} \beta g(y)/2 < \inf_{y\in\sigma} \beta g(y) \le \beta g(\hat\sigma). \end{aligned}$$

Hence there is an arc $\gamma(m, j) : \mathbf{I} \to X$ from some point of $g(\hat\sigma)$ to $z(v_m, j)$ of $\operatorname{diam}_d \gamma(m, j)(\mathbf{I}) < \alpha g(\hat\sigma)/2$ by Lemma 5.3. Define

$$h(\hat\sigma) = g(\hat\sigma) \cup \{z(v_m, j) \mid m = 1, 2 \text{ and } j = 0, \cdots, k\}.$$

Note that $\operatorname{card} h(\hat\sigma) \ge k + 1$. Moreover, $d_H(g(\hat\sigma), h(\hat\sigma)) \le \beta g(\hat\sigma) \le \alpha g(\hat\sigma)/2$. Let $\phi : \mathbf{I} \to \operatorname{Fin}(X)$ be a map defined by

$$\phi(t) = g(\hat\sigma) \cup \{\gamma(m, j)(t) \mid m = 1, 2 \text{ and } j = 0, \cdots, k\},$$

which is a path from $g(\hat\sigma)$ to $h(\hat\sigma)$. For each $m = 1, 2$, define a map $h : \langle v_m, \hat\sigma\rangle \to \operatorname{Fin}(X)$ of the segment between v_m and $\hat\sigma$ in σ as follows:

$$h((1 - t)v_m + t\hat\sigma) = \begin{cases} g((1 - 2t)v_m + 2t\hat\sigma) \cup \{z(v_m, j) \mid j = 0, \cdots, k\} & \text{if } 0 \le t \le 1/2, \\ \phi(2t - 1) \cup \{z(v_m, j) \mid j = 0, \cdots, k\} & \text{if } 1/2 \le t \le 1. \end{cases}$$

Then for every $y \in \sigma$, when $y = (1 - t)v_m + t\hat\sigma$, $0 \le t \le 1/2$,

$$\begin{aligned} d_H(g(\hat\sigma), h(y)) &\le \max\{d_H(g(\hat\sigma), g((1 - 2t)v_m + 2t\hat\sigma)), \max\{d(z(v_m, j), g(\hat\sigma)) \mid j = 0, \cdots, k\}\} \\ &\le \max\{\operatorname{diam}_{d_H} g(\sigma), \beta g(\hat\sigma)\} \le \max\{\inf_{y'\in\sigma} \beta g(y')/2, \beta g(\hat\sigma)\} \le \beta g(\hat\sigma) \le \alpha g(\hat\sigma)/2, \end{aligned}$$

and when $y = (1 - t)v_m + t\hat\sigma$, $1/2 \le t \le 1$,

$$\begin{aligned} d_H(g(\hat\sigma), h(y)) &\le \max\{d_H(g(\hat\sigma), \phi(2t - 1)), \max\{d(z(v_m, j), g(\hat\sigma)) \mid j = 0, \cdots, k\}\} \\ &\le \max\{\operatorname{diam}_d \gamma(n, j)(\mathbf{I}), \beta g(\hat\sigma) \mid n = 1, 2 \text{ and } j = 0, \cdots, k\} \\ &\le \max\{\alpha g(\hat\sigma)/2, \beta g(\hat\sigma)\} = \alpha g(\hat\sigma)/2. \end{aligned}$$

Hence, due to condition (3), we have

$$\begin{aligned} d_H(g(y), h(y)) &\le d_H(g(y), g(\hat\sigma)) + d_H(g(\hat\sigma), h(y)) \le \operatorname{diam}_{d_H} g(\sigma) + \alpha g(\hat\sigma)/2 \\ &< \inf_{y'\in\sigma} \beta g(y')/2 + \alpha g(\hat\sigma)/2 \le \beta g(\hat\sigma)/2 + \alpha g(\hat\sigma)/2 \le 3\alpha g(\hat\sigma)/4 \\ &\le 3 \sup_{y'\in\sigma} \alpha g(y')/4 < \inf_{y'\in\sigma} \alpha g(y') \le \alpha g(y). \end{aligned}$$

Note that for each $y \in \sigma$, $h(y)$ contains $\{z(v_1, j) \mid j = 0, \cdots, k\}$ or $\{z(v_2, j) \mid j = 0, \cdots, k\}$, so $\operatorname{card} h(y) \geq k + 1$.

By induction, we shall construct a map $h : |K| \to \operatorname{Fin}(X)$ such that for each $y \in \sigma \in K \setminus K^{(0)}$, $h(y) = \bigcup_{a \in A(y)} h(a)$ for some $A(y) \in \operatorname{Fin}(|\sigma^{(1)}|)$. Assume that h extends over $|K^{(n)}|$ for some $n < \omega$ such that for every $y \in \sigma \in K^{(n)} \setminus K^{(0)}$, $h(y) = \bigcup_{a \in A(y)} h(a)$ for some $A(y) \in \operatorname{Fin}(|\sigma^{(1)}|)$. Take any $\sigma \in K^{(n+1)} \setminus K^{(n)}$. By Lemma 3.3 of [4], there exists a map $r : \sigma \to \operatorname{Fin}(\partial\sigma)$ such that $r(y) = \{y\}$ for all $y \in \partial\sigma$, where $\partial\sigma$ is the boundary of σ. The map $h|_{\partial\sigma}$ induces $\tilde{h} : \operatorname{Fin}(\partial\sigma) \to \operatorname{Fin}(X)$ defined by $\tilde{h}(A) = \bigcup_{a \in A} h(a)$. Then we can obtain the composition $h_\sigma = \tilde{h}r : \sigma \to \operatorname{Fin}(X)$. It follows from the definition that $h_\sigma|_{\partial\sigma} = h|_{\partial\sigma}$. Observe that for each $y \in \sigma$,

$$h_\sigma(y) = \tilde{h}r(y) = \bigcup_{y' \in r(y)} h(y') = \bigcup_{y' \in r(y)} \bigcup_{a \in A(y')} h(a) = \bigcup_{a \in \bigcup_{y' \in r(y)} A(y')} h(a),$$

where $h(y') = \bigcup_{a \in A(y')} h(a)$ for some $A(y') \in \operatorname{Fin}(|\sigma^{(1)}|)$ by the inductive assumption. Thus we can extend h over $|K^{(n+1)}|$ by $h|_\sigma = h_\sigma$ for all $\sigma \in K^{(n+1)} \setminus K^{(n)}$.

After completing this induction, we can obtain a map $h : |K| \to \operatorname{Fin}(X)$. For each $\sigma \in K \setminus K^{(0)}$, each $y \in \sigma$ and each $a \in |\sigma^{(1)}|$, we get

$$\begin{aligned} d_H(g(y), h(a)) &\leq d_H(g(y), g(a)) + d_H(g(a), h(a)) < \operatorname{diam}_{d_H} g(\sigma) + \alpha g(a) \\ &< \inf_{y' \in \sigma} \beta g(y')/2 + \sup_{y' \in \sigma} \alpha g(y') \leq \inf_{y' \in \sigma} \alpha g(y')/4 + 4 \inf_{y' \in \sigma} \alpha g(y')/3 \\ &= 19 \inf_{y' \in \sigma} \alpha g(y')/12 < 2\alpha g(y). \end{aligned}$$

Therefore we have

$$d_H(g(y), h(y)) = d_H\left(g(y), \bigcup_{a \in \bigcup_{y' \in r(y)} A(y')} h(a)\right) \leq \max_{a \in \bigcup_{y' \in r(y)} A(y')} d_H(g(y), h(a)) < 2\alpha g(y),$$

which implies that h is $\mathcal{V}$-close to g. Remark that $\{z(v, j) \mid j = 0, \cdots, k\} \subset h(y)$ for some $v \in \sigma^{(0)}$, and hence $\operatorname{card} h(y) \geq k + 1$. It follows that $h(|K|) \cap \operatorname{Fin}^k(X) = \emptyset$. Then we may replace $h(y)$ with $g(y) \cup h(y)$ for every $y \in |K|$, so we have $g(y) \subset h(y)$. The rest of this proof is to show that $\operatorname{cl} h(|K|) \cap \operatorname{Fin}^k(X) = \emptyset$.

Suppose that there exists a sequence $\{y_n\}_{n<\omega}$ of $|K|$ such that $\{h(y_n)\}_{n<\omega}$ converges to some $A \in \operatorname{Fin}^k(X)$. Take the carrier $\sigma_n \in K$ of y_n and choose $v_n \in \sigma_n^{(0)}$ so that $\{z(v_n, j) \mid j = 0, \cdots, k\} \subset h(y_n)$. Since $g(y_n) \subset h(y_n)$, replacing $\{g(y_n)\}_{n<\omega}$ with a subsequence, we can obtain $B \subset A$ to which $\{g(y_n)\}_{n<\omega}$ converges by Lemma 5.2. Then $\{\beta g(y_n)\}_{n<\omega}$ converges to $\beta(B) > 0$. On the other hand, for every $\epsilon > 0$, there exists $n_0 < \omega$ such that if $n \geq n_0$, then $d_H(h(y_n), A) < \epsilon$. Then we can choose $0 \leq i(n) < j(n) \leq k$ for each $n \geq n_0$ so that $z(v_n, i(n)), z(v_n, j(n)) \in B_d(a, \epsilon)$ for some $a \in A$ because

$$\operatorname{card} A \leq k < k + 1 = \operatorname{card}\{z(v_n, j) \mid j = 0, \cdots, k\}.$$

Note that

$$\begin{aligned} \beta g(y_n)/(8(k+1)) &\leq \sup_{y \in \sigma_n} \beta g(y)/(8(k+1)) < \inf_{y \in \sigma_n} \beta g(y)/(4(k+1)) \leq \beta g(v_n)/(4(k+1)) \\ &\leq d(z(v_n, i(n)), z(v_n, j(n))) < 2\epsilon, \end{aligned}$$

which means that $\{\beta g(y_n)\}_{n<\omega}$ converges to 0. This is a contradiction. Consequently, $\operatorname{cl} h(|K|) \cap \operatorname{Fin}^k(X) = \emptyset$. □

Combining Lemma 5.4 with Proposition 5.5, we can obtain the following:

Proposition 5.6. *Let X be a non-degenerate, connected and locally path-connected space. Then every compact subset of* $\mathrm{Fin}(X)$ *is a strong Z-set.*

6. The κ-discrete n-cells property of $\mathrm{Fin}(X)$

This section is devoted to detecing the κ-discrete n-cells property in $\mathrm{Fin}(X)$. First, we show the following lemma:

Lemma 6.1. *Let X be a locally path-connected space. Suppose that any neighborhood of each point in X contains a discrete subset of cardinality $\geq \kappa$. Then the hyperspace* $\mathrm{Fin}(X)$ *satisfies the following:*

- *Let K_γ be a simplicial complex, $\gamma < \kappa$. For each open cover $\mathcal{V}$ of* $\mathrm{Fin}(X)$*, and each map $g : \bigoplus_{\gamma<\kappa} |K_\gamma| \to \mathrm{Fin}(X)$, there exists a map $h : \bigoplus_{\gamma<\kappa} |K_\gamma| \to \mathrm{Fin}(X)$ such that h is $\mathcal{V}$-close to g and the family $\{h(|K_\gamma|)\}_{\gamma<\kappa}$ is locally finite in* $\mathrm{Fin}(X)$.

Proof. Fix any admissible metric d on X. Let $\alpha, \beta : \mathrm{Fin}(X) \to (0, 1)$ be the same maps and replace each K_γ, $\gamma < \kappa$, with the same subdivision as in Proposition 5.5. For each $n < \omega$, we can find a locally finite open cover $\mathcal{V}_n$ of X of mesh $< 1/n$. By the assumption, for every $n < \omega$, each non-empty $V \in \mathcal{V}_n$ contains a discrete subset $Z(V) = \{z_V^\gamma\}_{\gamma<\kappa}$ of cardinality κ. Let $Z^\gamma(n) = \{z_V^\gamma \mid \emptyset \neq V \in \mathcal{V}_n\}$, $\gamma < \kappa$. Here we may assume that $Z^\gamma(n) \cap Z^\tau(n) = \emptyset$ if $\gamma \neq \tau$. Indeed, we shall show it by transfinite induction. Suppose that for some $\gamma < \kappa$, $Z^\tau(n) \cap Z^{\tau'}(n) = \emptyset$ if $\tau < \tau' < \gamma$. By the local finiteness of $\mathcal{V}_n$, for every $z_V^\gamma \in Z^\gamma(n)$, $\{V' \in \mathcal{V}_n \mid z_V^\gamma \in V'\}$ is finite. Since X has no isolated points and each $Z(V)$ is discrete, we can find a point $x_V^\gamma \in V$ sufficiently close to z_V^γ such that $x_V^\gamma \notin \bigcup_{\tau<\gamma} Z^\tau(n)$ and even if z_V^γ is substituted by x_V^γ, $Z(V)$ is still discrete. Due to this substitution, we have that $Z^\tau(n) \cap Z^\gamma(n) = \emptyset$ for all $\tau < \gamma$. Put $Z(n) = \bigcup_{\emptyset\neq V\in\mathcal{V}_n} Z(V) = \bigoplus_{\gamma<\kappa} Z^\gamma(n)$, that is locally finite in X.

To begin with, we shall construct the restriction $h|_{K_\gamma^{(0)}}$, $\gamma < \kappa$. For every $v \in K_\gamma^{(0)}$, there is $n_v \geq 2$ such that $1/n_v < \beta g(v)/4 \leq 1/(n_v - 1)$. Then we can find a point $z^\gamma(v) \in Z^\gamma(n_v) \subset Z(n_v)$, $\gamma < \kappa$, so that $d(z^\gamma(v), g(v)) < 1/n_v$. Remark that for any $u \in K_\tau^{(0)}$ and $u' \in K_{\tau'}^{(0)}$ with $n_u = n_{u'}$, $z^\tau(u) \neq z^{\tau'}(u')$ if $\tau < \tau' < \kappa$. Let $h(v) = g(v) \cup \{z^\gamma(v)\} \in \mathrm{Fin}(X)$. Note that

$$d_H(g(v), h(v)) < 1/n_v < \beta g(v)/4 \leq \alpha g(v)/2.$$

After the same construction of map as in Proposition 5.5, we can obtain a map $h : \bigoplus_{\gamma<\kappa} |K_\gamma| \to \mathrm{Fin}(X)$ so that h is $\mathcal{V}$-close to g and for any $\gamma < \kappa$ and any $y \in |K_\gamma|$, $h(y)$ contains a point $z^\gamma(v) \in Z(n_v)$ for some $v \in \sigma^{(0)}$, where $\sigma \in K_\gamma$ is the carrier of y. Here we may replace $h(y)$ with the union $g(y) \cup h(y)$ for every $y \in |K_\gamma|$, so we have $g(y) \subset h(y)$. It remains to show that $\{h(|K_\gamma|)\}_{\gamma<\kappa}$ is locally finite in $\mathrm{Fin}(X)$.

Suppose the contrary, so we can find a sequence $\{\gamma_i\}_{i<\omega}$ of κ such that $\gamma_i \neq \gamma_j$ if $i \neq j$, and $\{h(y_{\gamma_i})\}_{i<\omega}$, $y_{\gamma_i} \in |K_{\gamma_i}|$, converges to some $A \in \mathrm{Fin}(X)$. For the sake of convenience, replace each γ_i with i. Let $\sigma_i \in K_i$ be the carrier of y_i and choose a vertex $v_i \in \sigma_i^{(0)}$ so that $z^i(v_i) \in h(y_i)$. Since $g(y_i) \subset h(y_i)$, replacing $\{g(y_i)\}_{i<\omega}$ with a subsequence, we can obtain a subset $B \subset A$ to which $\{g(y_i)\}_{i<\omega}$ converges by Lemma 5.2. Then $\{\beta g(y_i)\}_{i<\omega}$ is converging to $\beta(B) > 0$. On the other hand, any subsequence of $\{z^i(v_i)\}_{i<\omega}$ has an accumulation point in A because $\{h(y_i)\}_{i<\omega}$ converges to A. Observe that $\{n_{v_i}\}_{i<\omega}$ diverges to ∞. Indeed, supposing the converse, we can find $n_0 < \omega$ and replace $\{n_{v_i}\}_{i<\omega}$ with a subsequence so that $n_{v_i} = n_0$ for all $i < \omega$. By the choice of $z^i(v_i)$, $\{z^i(v_i)\}_{i<\omega}$ is pairwise distinct and contained in the locally finite subset $Z(n_0)$, which is a contradiction. Hence $\{\beta g(v_i)\}_{i<\omega}$ converges to 0 since $\beta g(v_i)/4 \leq 1/(n_{v_i} - 1)$. Moreover, it follows

from condition (2) of Proposition 5.5 that for every $i < \omega$,

$$\beta g(y_i) \leq \sup_{y \in \sigma_i} \beta g(y) < 2 \inf_{y \in \sigma_i} \beta g(y) \leq 2\beta g(v_i).$$

Therefore $\{\beta g(y_i)\}_{i<\omega}$ converges to 0, which is a contradiction. Thus we conclude that the family $\{h(|K_\gamma|)\}_{\gamma<\kappa}$ is locally finite in $\mathrm{Fin}(X)$. $\square$

It is said that a space X has *the countable locally finite approximation property* provided that for every open cover $\mathcal{U}$ of X, there exists a sequence $\{f_n : X \to X\}_{n<\omega}$ of maps such that each f_n is $\mathcal{U}$-close to the identity map on X and the family $\{f_n(X)\}_{n<\omega}$ is locally finite in X.

Proposition 6.2. *Let X be a locally path-connected and nowhere locally compact space. Then* $\mathrm{Fin}(X)$ *has the countable locally finite approximation property.*

Proof. By the same argument as Proposition 5.5, we need only to show that for any simplicial complex K, any map $g : |K| \to \mathrm{Fin}(X)$ and any open cover $\mathcal{V}$ of $\mathrm{Fin}(X)$, there exists a map $g_i : |K| \to \mathrm{Fin}(X)$ for each $i < \omega$ such that each g_i is $\mathcal{V}$-close to g and $\{g_i(|K|)\}_{i<\omega}$ is locally finite in $\mathrm{Fin}(X)$. Since X is nowhere locally compact, it satisfies the assumption as in Lemma 6.1 with respect to $\kappa = \omega$. Hence this lemma guarantees the countable locally finite approximation property of $\mathrm{Fin}(X)$. $\square$

The next two lemmas are useful for recognizing the κ-discrete n-cells property in a space.

Lemma 6.3 (Lemma 3.1 of [1])**.** *Let $n < \omega$. A space X has the κ-discrete n-cells property if and only if the following condition holds:*

- *For each open cover $\mathcal{U}$ of X, and each map $f : \bigoplus_{\gamma<\kappa} A_\gamma \to X$, where each $A_\gamma = \mathbf{I}^n$, there is a map $g : \bigoplus_{\gamma<\kappa} A_\gamma \to X$ such that g is $\mathcal{U}$-close to f and the family $\{g(A_\gamma)\}_{\gamma<\kappa}$ is locally finite in X.*

Lemma 6.4 (Lemma 3.2 of [1])**.** *Let $n < \omega$. A space X with the countable locally finite approximation property has the κ-discrete n-cells property if and only if X has the λ-discrete n-cells property for any $\lambda \leq \kappa$ of uncountable cofinality.*

A subset A of a metric space $X = (X, d)$ is called *δ-discrete*, $\delta > 0$, provided that for any distinct points $a, a' \in A$, $d(a, a') \geq \delta$. The following lemma follows from the proof of [1, Lemma 6.2]:

Lemma 6.5. *Let X be a space and κ be of uncountable cofinality. For each subset $A \subset X$ of density $\geq \kappa$, it contains a discrete subset of cardinality $\geq \kappa$.*

Proof. Let $A \subset X$ be of density $\geq \kappa$. For each $n < \omega$, take any maximal 2^{-n}-discrete subset $D(n) \subset A$. If some $D(n)$ is of cardinality $\geq \kappa$, then the proof is finished. Conversely, suppose that every $D(n)$ is of cardinality $< \kappa$. Then $\bigcup_{n<\omega} D(n)$ is dense in A due to the maximality of $D(n)$. On the other hand, since κ is of uncountable cofinality, the cardinality of the countable union $\bigcup_{n<\omega} D(n)$ is less than κ, which contradicts to that A is of density $\geq \kappa$. Thus we conclude that A contains a discrete subset of cardinality $\geq \kappa$. $\square$

Now, we will prove the following:

Proposition 6.6. *Let X be a locally path-connected and nowhere locally compact space. If any neighborhood of each point in X is of density $\geq \kappa$, then the hyperspace* $\mathrm{Fin}(X)$ *has the κ-discrete n-cells property for every $n < \omega$.*

Proof. The hyperspace $\mathrm{Fin}(X)$ has the countable locally finite approximation property by Proposition 6.2. Hence we may assume that κ is of uncountable cofinality due to Lemma 6.4. Let $n < \omega$ and take any open cover $\mathcal{V}$ of $\mathrm{Fin}(X)$. By virtue of Lemma 6.3, it suffices to prove that for any map $g : \bigoplus_{\gamma<\kappa} A_\gamma \to \mathrm{Fin}(X)$, where each $A_\gamma = \mathbf{I}^n$, there exists a $\mathcal{V}$-close map $h : \bigoplus_{\gamma<\kappa} A_\gamma \to \mathrm{Fin}(X)$ to g such that $\{h(A_\gamma)\}_{\gamma<\kappa}$ is locally finite in $\mathrm{Fin}(X)$. It follows from Lemma 6.5 that any neighborhood of each point in X contains a discrete subset of cardinality $\geq \kappa$. Using Lemma 6.1, we can obtain the desired map h. Thus the proof is completed. □

7. Proof of the main theorem

In this final section, applying the characterization 2.1, we prove the main theorem.

Proof of the main theorem. The separable case follows from Theorem 1.1, so we only consider κ be uncountable.

(The "only if" part) According to Propositions 3.1 and 3.3, the hyperspace $\mathrm{Fin}(X)$ is a strongly countable-dimensional and σ-locally compact AR of density κ. It follows from Proposition 4.2 that X is strongly universal for the class of finite-dimensional compact metrizable spaces. Due to Proposition 5.6, every finite-dimensional compact subset of $\mathrm{Fin}(X)$ is a strong Z-set in $\mathrm{Fin}(X)$. Moreover, the κ-discrete n-cells property of $\mathrm{Fin}(X)$ for every $n < \omega$ follows from Proposition 6.6. Using Theorem 2.1, we conclude that $\mathrm{Fin}(X)$ is homeomorphic to $\ell_2^f(\kappa)$.

(The "if" part) Since $\mathrm{Fin}(X)$ is homeomorphic to $\ell_2^f(\kappa)$, it is a strongly countable-dimensional and σ-locally compact AR, and hence the space X is connected, locally path-connected, strongly countable-dimensional and σ-locally compact by Propositions 3.1 and 3.3. Remark that for each $A \in \mathrm{Fin}(X)$, all neighborhoods of A are of density κ. It follows from Proposition 3.2 that any neighborhood of each point in X is also of density κ. The proof is complete. □

References

[1] T. Banakh and I. Zarichnyy, *Topological groups and convex sets homeomorphic to non-separable Hilbert spaces*, Cent. Eur. J. Math. **6**:1 (2008), 77–86.

[2] D.W. Curtis, *Hyperspaces homeomorphic to Hilbert space*, Proc. Amer. Math. Soc. **75**, (1979), 126–130.

[3] D.W. Curtis, *Hyperspaces of finite subsets as boundary sets*, Topology Appl. **22**, (1986), 97–107.

[4] D.W. Curtis and N.T. Nhu, *Hyperspaces of finite subsets which are homeomorphic to $\aleph_0$-dimensional linear metric spaces*, Topology Appl. **19**, (1985), 251–260.

[5] K. Koshino, *Characterizing non-separable sigma-locally compact infinite-dimensional manifolds and its applications*, J. Math. Soc. Japan **66** (2014), 1155–1189.

[6] J. van Mill, The infinite-dimensional topology of function spaces, North-Holland Math. Library, **64**, North-Holland Publishing Co., Amsterdam, 2001.

[7] K. Mine, K. Sakai and M. Yaguchi, *Hyperspaces of finite sets in universal spaces for absolute Borel classes*, Bull. Pol. Acad. Sci. Math. **53**, (2005), 409–419.

[8] K. Sakai, Geometric Aspects of General Topology, Springer, SMM, Springer, Tokyo, 2013.

[9] M. Yaguchi, *Hyperspaces of finite subsets of non-separable Hilbert spaces*, Tsukuba J. Math. **30**, (2006), 181–193.

Faculty of Engineering, Kanagawa University, Yokohama, 221-8686, Japan
E-mail address: pt120401we@kanagawa-u.ac.jp